Integration

$$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$$

Example:

$$\int e^x (\sin x + \cos x)dx = e^x.\sin x + c$$

$$\int e^x (\tan x + \sec^2 x)\, dx = e^x \tan x + c$$

$$\int e^x (x^2 + 2x + 1)\, dx = e^x (x^2 + 1) + c$$

Reverse Process of Differentiation

Evaluate the following:

1. $I = \int x^2.e^x\, dx$

$= (x^2\int e^x\, dx) - \int\left[\frac{d}{dx}(x^2).\int e^x\, dx\right]dx$

$= x^2.\, e^x - \int 2x\, .\, e^x\, .\, dx$

$= x^2\, e^x - 2\int x\, .\, e^x\, dx$

$= x^2\, e^x - 2\left[x\int e^x\, dx - \int\left[\frac{d}{dx}(x).\int e^x\, dx\right]dx\right]$

$= x^2\, e^x - 2\left[x\, e^x - \int 1\, .\, e^x\, dx\right]$

$= x^2\, e^x - 2\, (x\, e^x - e^x)$

$= e^x\, (x^2 - 2x + 2)$

★ Bracket Method (For Verification)

$\int x^2\, e^x\, dx = (x^2e^x) - (2x\, e^x) + (2.e^x)$

$= e^x\, (x^2 - 2x + 2)$

2. $I = \int x^3\, e^{ax}\, dx$

$= \left[x^3\int e^{ax}\, dx\right] - \int\left[\frac{d}{dx}(x^3)\, .\int e^{ax}\, dx\right]dx$

$= \left(x^3\, .\, \frac{e^{ax}}{a}\right) - \int 3x^2.\frac{e^{ax}}{a}\, .\, dx$

$= \frac{x^3\, e^{ax}}{a} - \frac{3}{a}\int x^2\, .\, e^{ax}\, dx$

$= \frac{x^3\, e^{ax}}{a} - \frac{3}{a}\left[\left(x^2\, .\int e^{ax}\, dx\right)\right.$

$\left. -\int\left[\frac{d}{dx}(x^2)\, .\int e^{ax}\, dx\right]dx\right]$

$= \frac{x^3\, e^{ax}}{a} - \frac{3}{a}\left[x^2\, .\frac{e^{ax}}{a} - \int 2x\, .\frac{e^{ax}}{a}\, .\, dx\right]$

$= \frac{x^3\, e^{ax}}{a} - \frac{3x^2\, e^{ax}}{a^2} + \frac{6}{a^2}\int x\, .\, e^{ax}\, dx$

$= \frac{x^3\, e^{ax}}{a} - \frac{3x^2\, e^{ax}}{a^2} +$

$\frac{6}{a^2}\left[x\int e^{ax}\, dx - \int\left[\frac{d}{dx}(x).\int e^{ax}\, dx\right]dx\right]$

$= \frac{x^3\, e^{ax}}{a} - \frac{3x^2\, e^{ax}}{a^2} +$

$\frac{6}{a^2}\left[x\, .\frac{e^{ax}}{a} - \int 1\, .\frac{e^{ax}}{a}\, dx\right]$

$= \frac{x^3\, e^{ax}}{a} - \frac{3x^2\, e^{ax}}{a^2} + \frac{6x\, e^{ax}}{a^3} - \frac{6}{a^3}\int e^{ax}\, dx$

$= \frac{x^3\, e^{ax}}{a} - \frac{3x^2\, e^{ax}}{a^2} + \frac{6x\, e^{ax}}{a^3} - \frac{6}{a^3}\left(\frac{e^{ax}}{a}\right)$

$= \frac{x^3\, e^{ax}}{a} - \frac{3x^2\, e^{ax}}{a^2} + \frac{6x\, e^{ax}}{a^3} - \frac{6\, e^{ax}}{a^4}$

$= e^{ax}\left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right)$

★ Bracket Method (Verification Method)

$\int x^3\, e^{ax}\, dx = \left(x^3\, .\frac{e^{ax}}{a}\right) - \left(3x^2\, .\frac{e^{ax}}{a^2}\right)$

$+ \left(6x\, .\frac{e^{ax}}{a^3}\right) - \left(6\, .\frac{e^{ax}}{a^4}\right)$

$= e^{ax}\left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right)$

3. $I = \int x \sin x\, dx$

$= \left(x\int \sin x\, dx\right) - \int\left[\frac{d}{dx}(x)\, .\int \sin x\, dx\right]dx$

$= x\, (-\cos x) - \int 1(-\cos x)dx$

$= -x \cos x - (-\sin x)$

$= \sin x - x \cos x$

★ Bracket Method:

$\int x \sin x\, dx = (x.\, (-\cos x) - (1 - \sin x)$

$= \sin x - x \cos X$

4. $\int x \cos x\, dx = \left(x\int \cos x\, dx\right) -$

$\int\left[\frac{d}{dx}(x).\int \cos x\, dx\right]dx$

$= x\, .\, \sin x - \int (1.\, \sin x)\, dx$

$= x \sin x - (-\cos x)$

$= \cos x + x \sin x$

★ Bracket method:

$\int x \cos x\, dx = (x.\, \sin x) - (1.\, -\cos x)$

$= \cos x + x \sin x$

Observe bracket method for the following (For Verification)

5. $\int x^2 \sin x\, dx = [x^2\, (-\cos x)] - [2x\, (-\sin x)]$ $+ [2.\, \cos x]$

$= 2 \cos x + 2x \sin x - x^2 \cos x.$

6) $\int x^2 \cos x\, dx = (x^2\, .\, \sin x) - (2x\, .\, -\cos x) + (2\, .\, -\sin x)$

$= x^2 \sin x + 2x \cos x - 2 \sin x.$

7) $\int x^3 \sin x\, dx = (x^3\, .\, -\cos x) - (3x^2\, .\, -\sin x)$ $+ (6x\, .\, \cos x) - (6\, .\, \sin x)$

$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$

8) $\int x^3 \cos x\, dx = (x^3.\sin x) - (3x^2.-\cos x)$ $+ (6x.\, -\sin x) - (6\, .\, -\cos x)$

$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x$

9) $\int x \sin^2 x\, dx = \int x\left(\frac{1 - \cos 2x}{2}\right)dx$

$= \int \frac{x}{2}\, dx - \frac{1}{2}\int x \cos 2x\, dx$

$= \frac{x^2}{4} - \frac{1}{2}\left[\left(x.\frac{\sin 2x}{2}\right) - \left(1.\frac{-\cos 2x}{4}\right)\right]$

$= \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8}$

10. $\int x \cos^2 x\, dx = \int x\left(\frac{1 + \cos 2x}{2}\right)dx$

$= \int \frac{x}{2}\, dx + \frac{1}{2}\int x \cos 2x\, dx$

$= \frac{x^2}{4} + \frac{1}{2}\left[\left(x.\frac{\sin 2x}{2}\right) - \left(1.\frac{-\cos 2x}{4}\right)\right]$

$= \frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8}$

11. $\int x \sec^2 x\, dx = (x.\tan x) - (1.\log|\sec x|)$

$= x \tan x - \log|\sec x|$

12. $\int x\, \text{cosec}^2 x\, dx = (x.-\cot x) - (1.-\log|\sin x|)$

$= -x \cot x + \log|\sin x|$

13. $\int x \tan^2 x\, dx = \int x\, (\sec^2 x - 1)\, dx$

$= \int x \sec^2 x\, dx - \int x\, dx$

$= x \tan x - \log|\sec x| - \frac{x^2}{2}$

14. $\int x \cot^2 x\, dx = \int x\, (\text{cosec}^2 x - 1)\, dx$

$= \int x\, \text{cosec}^2 x\, dx - \int x\, dx$

$= -x \cot x + \log|\sin x| - \frac{x^2}{2}$

15. $\int x \sin^{-1} x\, dx = \left[\sin^{-1} x.\int x\, dx\right] -$

$\int\left[\frac{d}{dx}(\sin^{-1} x).\int x\, dx\right]dx$

$= \sin^{-1} x\, .\frac{x^2}{2} - \int\frac{1}{\sqrt{1-x^2}}\, .\frac{x^2}{2}\, dx$

$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{2}\int\frac{1-(1-x^2)}{\sqrt{1-x^2}}\, dx$

$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}dx + \frac{1}{2}\int\sqrt{1-x^2}\, dx$

$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{2}\sin^{-1} x + \frac{1}{2}\, .\frac{1}{2}\left[x\sqrt{1-x^2} + \sin^{-1} x\right]$

$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{4}\sin^{-1} x + \frac{x\sqrt{1-x^2}}{4}$

16. $\int x \tan^{-1} x\, dx = \left[\tan^{-1} x\, .\int x\, dx\right] -$

$\int\left[\frac{d}{dx}(\tan^{-1} x).\int x\, dx\right]dx$

$= \tan^{-1} x\, .\frac{x^2}{2} - \int\frac{1}{1+x^2}\, .\frac{x^2}{2}\, dx$

$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int\left[\frac{(1+x^2) - 1}{1+x^2}\right]dx$

$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int 1\, dx + \frac{1}{2}\int\frac{1}{1+x^2}dx$

$= \frac{x^2 \tan^{-1} x}{2} - \frac{x}{2} + \frac{1}{2}(\tan^{-1} x)$

17. $\int \sin(\log x)\, dx = \int \sin(\log x).\, 1.dx$

$= \sin(\log x).\, x - \int \cos(\log x).\frac{1}{x}.\, x.\, dx$

$= x \sin(\log x) - \int \cos(\log x).1.dx$

$= x \sin(\log x) - \left[\cos(\log x).x + \int \sin(\log x)\, .\frac{1}{x}\, .\, x\, dx\right]$

$= x\sin(\log x) - x \cos(\log x) - \int \sin(\log x)\, dx$

$\therefore 2\int \sin(\log x)\, dx = x\left[\sin(\log x) - \cos(\log x)\right]$

$\therefore \int \sin(\log x)\, dx = \frac{x}{2}\left[\sin(\log x) - \cos(\log x)\right]$

18. $\int (\log x)^2\, dx = \int (\log x)^2.1.dx$

$= (\log x)^2.x - \int 2 \log x\, .\frac{1}{x}.\, x.\, dx$

$= x\, (\log x)^2 - 2\int \log x\, .\, 1\, .\, dx$

$= x\, (\log x)^2 - 2\left[\log x\, .\, x - \int\frac{1}{x}.x.dx\right]$

$= x\, (\log x)^2 - 2\left[x.\log x - \int 1\, dx\right]$

$= x\, (\log x)^2 - 2x \log x + 2(x)$

$= x\left[(\log x)^2 - 2 \log x + 2\right]$

19. $\int \sin\sqrt{x}\, dx$

put $\sqrt{x} = t$

$\frac{1}{2\sqrt{x}}\, dx = dt$

$\therefore dx = 2t\, .\, dt$

$\therefore I = \int \sin t.2t.dt$

$= 2\int \sin t.t.dt$

$= 2\, [\sin t - t \cos t]$

$= 2\, [\sin\sqrt{x} - \sqrt{x}\, .\cos\sqrt{x}]$

20. $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

$= \int 2 \tan^{-1} x\, dx$

$= 2\int \tan^{-1} x.1.dx$

$= 2\left[\tan^{-1} x.x - \int\frac{1}{1+x^2}.x.dx\right]$

$= 2x \tan^{-1} x - \int\frac{2x}{1+x^2}dx$

$= 2x \tan^{-1} x - \log|1 + x^2|$

21. $I = \int\frac{dx}{(x^2 + a^2)^2}$, $(a > 0, x \in R)$

Put $x = a \tan\theta$

$dx = a \sec^2\theta\, .\, d\theta$

$I = \int\frac{a \sec^2\theta\, .\, d\theta}{(a^2 \tan^2\theta + a^2)^2}$

$= \frac{1}{a^3}\int\frac{\sec^2\theta}{\sec^4\theta}d\theta$

$= \frac{1}{a^3}\int \cos^2\theta\, d\theta$

$= \frac{1}{a^3}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right] + c$

$= \frac{1}{a^3}\left[\frac{1}{2}.\, \tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{4}\sin\left[2 \tan^{-1}\left(\frac{x}{a}\right)\right]\right] + c$

22. $\int x^2\, .\cos x\, dx$

$= (x^2\, .\, \sin x) - (2x\, .\, -\cos x) + (2\, .\, -\sin x) + c$

$= x^2 \sin x + 2x \cos x - 2 \sin x + c$

$\int x^2\, .\, \sin x\, dx$

$= (x^2\, .\, -\cos x) - (2x\, .\, -\sin x) + (2\, .\, \cos x) + c$

$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$